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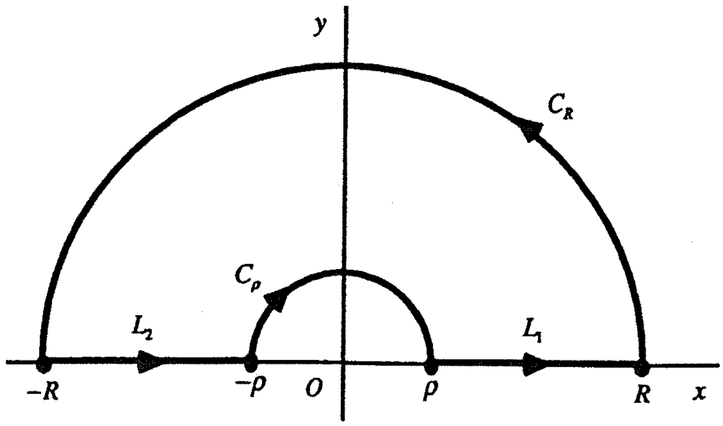
1. The main problem here is to derive the integration formula

$$\int_0^{\infty}\frac{\cos(ax)-\cos(bx)}{x^2}\,dx=\frac{\pi}{2}(b-a)\qquad (a\geq 0, b\geq 0),$$

using the indented contour shown below.



Applying the Cauchy-Goursat theorem to the function

$$f(z) = \frac{e^{iaz} - e^{ibz}}{z^2},$$

we have

$$\int_{L_1} f(z)\,dz + \int_{C_R} f(z)\,dz + \int_{L_2} f(z)\,dz + \int_{C_\rho} f(z)\,dz = 0,$$

or

$$\int_{L_1} f(z)\,dz + \int_{L_2} f(z)\,dz = -\int_{C_\rho} f(z)\,dz - \int_{C_R} f(z)\,dz.$$

Since L_1 and $-L_2$ have parametric representations

$$L_1 : z = re^{i0} = r \; (\rho \le r \le R) \quad \text{and} \quad -L_2 : z = re^{i\pi} = -r \; (\rho \le r \le R),$$

we can see that

$$\int_{L_1} f(z)\,dz + \int_{L_2} f(z)\,dz = \int_{L_1} f(z)\,dz - \int_{-L_2} f(z)\,dz = \int_\rho^R \frac{e^{iar} - e^{ibr}}{r^2}\,dr + \int_\rho^R \frac{e^{-iar} - e^{-ibr}}{r^2}\,dr$$

$$= \int_\rho^R \frac{(e^{iar} + e^{-iar}) - (e^{ibr} + e^{-ibr})}{r^2}\,dr = 2\int_\rho^R \frac{\cos(ar) - \cos(br)}{r^2}\,dr.$$

Thus

$$2\int_\rho^R \frac{\cos(ar) - \cos(br)}{r^2}\,dr = -\int_{C_\rho} f(z)\,dz - \int_{C_R} f(z)\,dz.$$

In order to find the limit of the first integral on the right here as $\rho \to 0$, we write

$$f(z) = \frac{1}{z^2}\left[\left(1 + \frac{iaz}{1!} + \frac{(iaz)^2}{2!} + \frac{(iaz)^3}{3!} + \cdots\right) - \left(1 + \frac{ibz}{1!} + \frac{(ibz)^2}{2!} + \frac{(ibz)^3}{3!} + \cdots\right)\right]$$

$$= \frac{i(a-b)}{z} + \cdots \quad (0 < |z| < \infty).$$

From this we see that $z = 0$ is a simple pole of $f(z)$, with ~~residue B_0~~ $= i(a-b)$. Thus

$$\lim_{\rho \to 0} \int_{C_\rho} f(z)\,dz = -B_0 \pi i = -i(a-b)\pi i = \pi(a-b).$$

As for the limit of the value of the second integral as $R \to \infty$, we note that if z is a point on C_R, then

$$f(z) \le \frac{|e^{iaz}|+|e^{ibz}|}{|z|^2} = \frac{e^{-ay}+e^{-by}}{R^2} \le \frac{1+1}{R^2} = \frac{2}{R^2}.$$

Consequently,

$$\left|\int_{C_R} f(z)\,dz\right| \le \frac{2}{R^2}\pi R = \frac{2\pi}{R} \to 0 \text{ as } R \to \infty.$$

It is now clear that letting $\rho \to 0$ and $R \to \infty$ yields

$$2\int_0^\infty \frac{\cos(ar)-\cos(br)}{r^2}\,dr = \pi(b-a).$$

This is the desired integration formula, with the variable of integration r instead of x. Observe that when $a=0$ and $b=2$, that result becomes

$$\int_0^\infty \frac{1-\cos(2x)}{x^2}\,dx = \pi.$$

But $\cos(2x) = 1-2\sin^2 x$, and we arrive at

$$\int_0^\infty \frac{\sin^2 x}{x^2}\,dx = \frac{\pi}{2}.$$

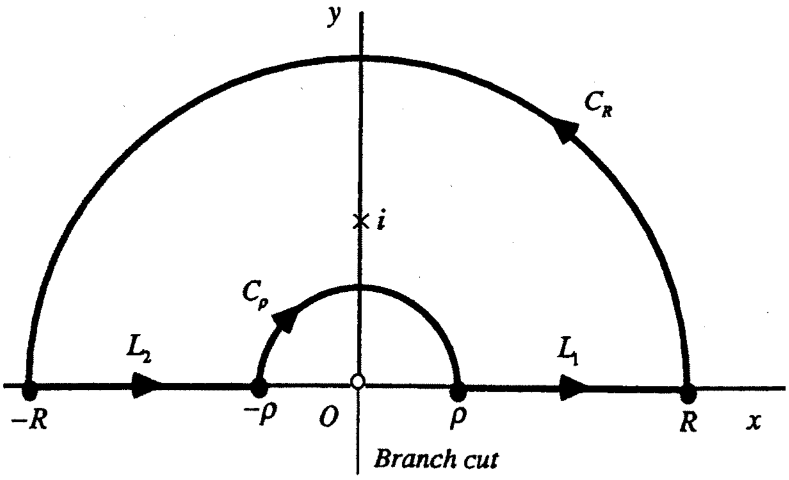
2

Let us first use the branch

$$f(z)=\frac{z^{-1/2}}{z^2+1}=\frac{\exp\left(-\frac{1}{2}\log z\right)}{z^2+1} \qquad \left(|z|>0, -\frac{\pi}{2}<\arg z<\frac{3\pi}{2}\right)$$

and the indented path shown below to evaluate the improper integral

$$\int_0^\infty \frac{dx}{\sqrt{x}(x^2+1)}.$$



Cauchy's residue theorem tells us that

$$\int_{L_1} f(z)\,dz+\int_{C_R} f(z)\,dz+\int_{L_2} f(z)\,dz+\int_{C_\rho} f(z)\,dz=2\pi i\operatorname*{Res}_{z=i} f(z),$$

or

$$\int_{L_1} f(z)\,dz+\int_{L_2} f(z)\,dz=2\pi i\operatorname*{Res}_{z=i} f(z)-\int_{C_\rho} f(z)\,dz-\int_{C_R} f(z)\,dz.$$

Since

$$L_1: z=re^{i0}=r\ (\rho\le r\le R) \quad \text{and} \quad -L_2: z=re^{i\pi}=-r\ (\rho\le r\le R),$$

we may write

$$\int_{L_1} f(z)dz+\int_{L_2} f(z)dz=\int_\rho^R\frac{dr}{\sqrt{r}(r^2+1)}-i\int_\rho^R\frac{dr}{\sqrt{r}(r^2+1)}=(1-i)\int_\rho^R\frac{dr}{\sqrt{r}(r^2+1)}.$$

Thus

$$(1-i)\int_\rho^R\frac{dr}{\sqrt{r}(r^2+1)}=2\pi i\operatorname*{Res}_{z=i} f(z)-\int_{C_\rho} f(z)\,dz-\int_{C_R} f(z)\,dz.$$

Now the point $z=i$ is evidently a simple pole of $f(z)$, with residue

$$\operatorname*{Res}_{z=i} f(z)=\left.\frac{z^{-1/2}}{z+i}\right]_{z=i}=\frac{\exp\left[-\frac{1}{2}\log i\right]}{2i}=\frac{\exp\left[-\frac{1}{2}\left(\ln 1+i\frac{\pi}{2}\right)\right]}{2i}=\frac{e^{-i\pi/4}}{2i}=\frac{1}{2i}\left(\frac{1-i}{\sqrt{2}}\right).$$

Furthermore,

$$\left|\int_{C_\rho} f(z)dz\right|\le\frac{\pi\rho}{\sqrt{\rho}(1-\rho^2)}=\frac{\pi\sqrt{\rho}}{1-\rho^2}\to 0 \text{ as } \rho\to 0$$

and

$$\left|\int_{C_R} f(z)dz\right|\le\frac{\pi\sqrt{R}}{(R^2-1)}=\frac{\pi}{\sqrt{R}\left(R-\frac{1}{R}\right)}\to 0 \text{ as } R\to\infty.$$

Finally, then, we have

$$(1-i)\int_0^\infty\frac{dr}{\sqrt{r}(r^2+1)}=\frac{\pi(1-i)}{\sqrt{2}},$$

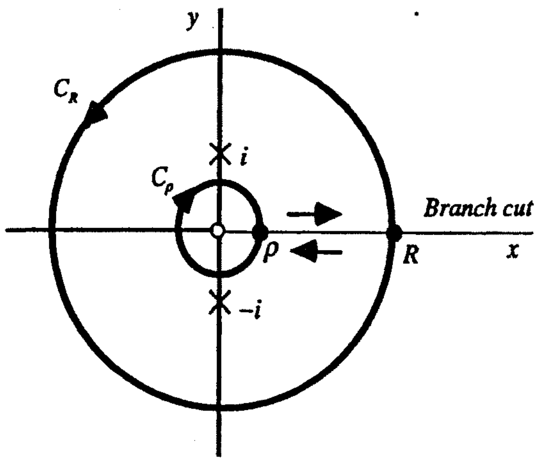
which is the same as

$$\int_0^\infty\frac{dx}{\sqrt{x}(x^2+1)}=\frac{\pi}{\sqrt{2}}.$$

3 To evaluate the improper integral $\int_0^\infty\frac{dx}{\sqrt{x}(x^2+1)}$, we now use the branch

$$f(z)=\frac{z^{-1/2}}{z^2+1}=\frac{\exp\left(-\frac{1}{2}\log z\right)}{z^2+1}\qquad (|z|>0,\ 0<\arg z<2\pi)$$

and the simple closed contour shown in the figure below, which is similar to Fig. 99 in Sec. 77. We stipulate that $\rho<1$ and $R>1$, so that the singularities $z=\pm i$ are between C_ρ and C_R.



Since a parametric representation for the upper edge of the branch cut from ρ to R is $z = re^{i0}$ $(\rho \le r \le R)$, the value of the integral of f along that edge is

$$\int_\rho^R \frac{\exp\left[-\frac{1}{2}(\ln r + i0)\right]}{r^2+1}dr = \int_\rho^R \frac{1}{\sqrt{r}(r^2+1)}dr.$$

A representation for the lower edge from ρ to is R is $z = re^{i2\pi}$ $(\rho \le r \le R)$, and so the value of the integral of f along that edge from R to ρ is

$$-\int_\rho^R \frac{\exp\left[-\frac{1}{2}(\ln r + i2\pi)\right]}{r^2+1}dr = -e^{-i\pi}\int_\rho^R \frac{1}{\sqrt{r}(r^2+1)}dr = \int_\rho^R \frac{1}{\sqrt{r}(r^2+1)}dr.$$

Hence, by the residue theorem,

$$\int_\rho^R \frac{1}{\sqrt{r}(r^2+1)}dr + \int_{C_R} f(z)dz + \int_\rho^R \frac{1}{\sqrt{r}(r^2+1)}dr + \int_{C_\rho} f(z)dz = 2\pi i(B_1 + B_2),$$

where

$$B_1 = \operatorname*{Res}_{z=i} f(z) = \left.\frac{z^{-1/2}}{z+i}\right]_{z=i} = \frac{\exp\left[-\frac{1}{2}\log i\right]}{2i} = \frac{\exp\left[-\frac{1}{2}\left(\ln 1 + i\frac{\pi}{2}\right)\right]}{2i} = \frac{e^{-i\pi/4}}{2i}$$

and

$$B_2 = \operatorname*{Res}_{z=-i} f(z) = \left.\frac{z^{-1/2}}{z-i}\right]_{z=-i} = \frac{\exp\left[-\frac{1}{2}\log(-i)\right]}{-2i} = \frac{\exp\left[-\frac{1}{2}\left(\ln 1 + i\frac{3\pi}{2}\right)\right]}{-2i} = -\frac{e^{-i3\pi/4}}{2i}.$$

That is,

$$2\int_\rho^R \frac{1}{\sqrt{r}(r^2+1)}dr = \pi(e^{-i\pi/4} - e^{-i3\pi/4}) - \int_{C_\rho} f(z)dz - \int_{C_R} f(z)dz.$$

Since

$$\left|\int_{C_\rho} f(z)dz\right| \le \frac{2\pi\rho}{\sqrt{\rho}(1-\rho^2)} = \frac{2\pi\sqrt{\rho}}{1-\rho^2} \to 0 \text{ as } \rho \to 0$$

and

$$\left|\int_{C_R} f(z)dz\right| \le \frac{2\pi R}{\sqrt{R}(R^2-1)} = \frac{2\pi}{\sqrt{R}\left(R - \frac{1}{R}\right)} \to 0 \text{ as } R \to \infty,$$

we now find that

$$\int_0^\infty \frac{1}{\sqrt{r}(r^2+1)}dr = \pi\frac{e^{-i\pi/4} - e^{-i3\pi/4}}{2} = \pi\frac{e^{-i\pi/4} + e^{-i3\pi/4}e^{i\pi}}{2}$$

$$= \pi\frac{e^{i\pi/4} + e^{-i\pi/4}}{2} = \pi\cos\left(\frac{\pi}{4}\right) = \frac{\pi}{\sqrt{2}}.$$

When x, instead of r, is used as the variable of integration here, we have the desired result:

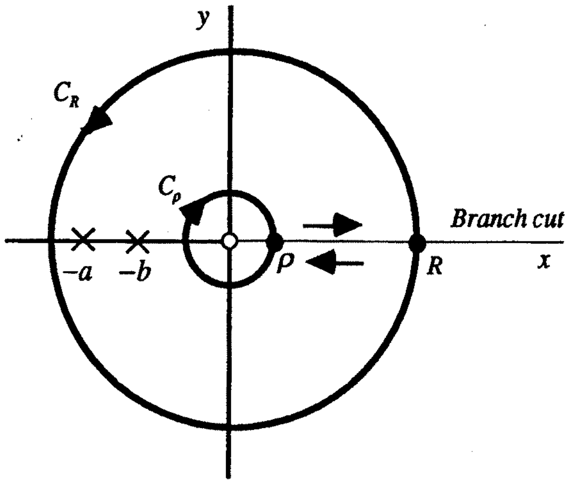
$$\int_0^\infty \frac{dx}{\sqrt{x}(x^2+1)} = \frac{\pi}{\sqrt{2}}.$$

(4)

Here we evaluate the integral $\int_0^\infty \frac{\sqrt[3]{x}}{(x+a)(x+b)}dx$, where $a > b > 0$. We consider the function

$$f(z) = \frac{z^{1/3}}{(z+a)(z+b)} = \frac{\exp\left(\frac{1}{3}\log z\right)}{(z+a)(z+b)} \qquad (|z| > 0, 0 < \arg z < 2\pi)$$

and the simple closed contour shown below, which is similar to the one used in Sec. 77. The numbers ρ and R are small and large enough, respectively, so that the points $z = -a$ and $z = -b$ are between the circles.



A parametric representation for the upper edge of the branch cut from ρ to R is $z = re^{i0}$ ($\rho \le r \le R$), and so the value of the integral of *f* along that edge is

$$\int_\rho^R \frac{\exp\left[\frac{1}{3}(\ln r + i0)\right]}{(r+a)(r+b)}dr = \int_\rho^R \frac{\sqrt[3]{r}}{(r+a)(r+b)}dr.$$

A representation for the lower edge from ρ to is R is $z = re^{i2\pi}$ ($\rho \le r \le R$). Hence the value of the integral of *f* along that edge from R to ρ is

$$-\int_\rho^R \frac{\exp\left[\frac{1}{3}(\ln r + i2\pi)\right]}{(r+a)(r+b)}dr = -e^{i2\pi/3}\int_\rho^R \frac{\sqrt[3]{r}}{(r+a)(r+b)}dr.$$

According to the residue theorem, then,

$$\int_\rho^R \frac{\sqrt[3]{r}}{(r+a)(r+b)}dr + \int_{C_R} f(z)dz - e^{i2\pi/3}\int_\rho^R \frac{\sqrt[3]{r}}{(r+a)(r+b)}dr + \int_{C_\rho} f(z)dz = 2\pi i(B_1 + B_2),$$

where

$$B_1 = \operatorname*{Res}_{z=-a} f(z) = \frac{\exp\left[\frac{1}{3}\log(-a)\right]}{-a+b} = -\frac{\exp\left[\frac{1}{3}(\ln a + i\pi)\right]}{a-b} = -\frac{e^{i\pi/3}\sqrt[3]{a}}{a-b}$$

and

$$B_2 = \operatorname*{Res}_{z=-b} f(z) = \frac{\exp\left[\frac{1}{3}\log(-b)\right]}{-b+a} = \frac{\exp\left[\frac{1}{3}(\ln b + i\pi)\right]}{-b+a} = \frac{e^{i\pi/3}\sqrt[3]{b}}{a-b}.$$

Consequently,

$$\left(1-e^{i2\pi/3}\right)\int_{\rho}^{R} \frac{\sqrt[3]{r}}{(r+a)(r+b)}\,dr = -\frac{2\pi i e^{i\pi/3}(\sqrt[3]{a}-\sqrt[3]{b})}{a-b} - \int_{C_\rho} f(z)dz - \int_{C_R} f(z)dz.$$

Now

$$\left|\int_{C_\rho} f(z)dz\right| \le \frac{\sqrt[3]{\rho}}{(a-\rho)(b-\rho)}2\pi\rho = \frac{2\pi\sqrt[3]{\rho}\,\rho}{(a-\rho)(b-\rho)} \to 0 \text{ as } \rho\to 0$$

and

$$\left|\int_{C_R} f(z)dz\right| \le \frac{\sqrt[3]{R}}{(R-a)(R-b)}2\pi R = \frac{2\pi R^2}{(R-a)(R-b)}\cdot\frac{1}{\sqrt[3]{R^2}} \to 0 \text{ as } R\to\infty.$$

Hence

$$\int_0^\infty \frac{\sqrt[3]{r}}{(r+a)(r+b)}\,dr = -\frac{2\pi i e^{i\pi/3}(\sqrt[3]{a}-\sqrt[3]{b})}{\left(1-e^{i2\pi/3}\right)(a-b)}\cdot\frac{e^{-i\pi/3}}{e^{-i\pi/3}} = \frac{2\pi i(\sqrt[3]{a}-\sqrt[3]{b})}{\left(e^{i\pi/3}-e^{-i\pi/3}\right)(a-b)}$$

$$= \frac{\pi(\sqrt[3]{a}-\sqrt[3]{b})}{\sin(\pi/3)(a-b)} = \frac{\pi(\sqrt[3]{a}-\sqrt[3]{b})}{\frac{\sqrt{3}}{2}(a-b)} = \frac{2\pi}{\sqrt{3}}\cdot\frac{\sqrt[3]{a}-\sqrt[3]{b}}{a-b}.$$

Replacing the variable of integration r here by x, we have the desired result:

$$\int_0^\infty \frac{\sqrt[3]{x}}{(x+a)(x+b)}\,dx = \frac{2\pi}{\sqrt{3}}\cdot\frac{\sqrt[3]{a}-\sqrt[3]{b}}{a-b} \qquad (a>b>0).$$

(6a) It's because the only singularity inside the contour is -1.

(6b) It's because there is no singularity inside the contour ($z=-1$ is not inside it).

(6c) Adding part (a) and (b) and taking $R \to \infty$, $\rho \to 0$

$$\left(1 - e^{-2a\pi i}\right) \int_0^\infty \frac{r^{-a}}{r+1} dr = 2\pi i \operatorname*{Res}_{z=-1} f_1(z)$$

$$\left(1 - \cos 2a\pi + i \sin 2a\pi\right) \int_0^\infty \frac{r^{-a}}{r+1} = 2\pi i \, e^{-ai\pi}$$

Thus $$\int_0^\infty \frac{r^{-a}}{r+1} dr = \frac{\pi}{\sin a\pi}.$$

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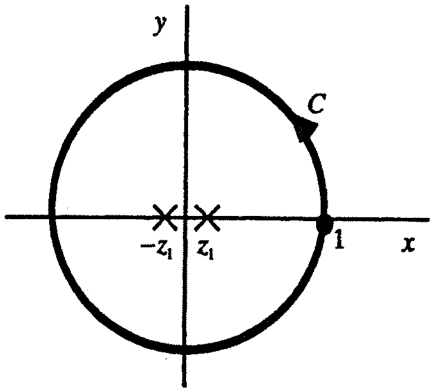
2. To evaluate the definite integral in question, write

$$\int_{-\pi}^{\pi}\frac{d\theta}{1+\sin^2\theta}=\int_C\frac{1}{1+\left(\dfrac{z-z^{-1}}{2i}\right)^2}\cdot\frac{dz}{iz}=\int_C\frac{4iz\,dz}{z^4-6z^2+1},$$

where C is the positively oriented unit circle $|z|=1$. This circle is shown below.



Solving the equation $(z^2)^2-6(z^2)+1=0$ for z^2 with the aid of the quadratic formula, we find that the zeros of the polynomial z^4-6z^2+1 are the numbers z such that $z^2=3\pm 2\sqrt{2}$. Those zeros are, then, $z=\pm\sqrt{3+2\sqrt{2}}$ and $z=\pm\sqrt{3-2\sqrt{2}}$. The first two of these zeros are exterior to the circle, and the second two are inside of it. So the singularities of the integrand in our contour integral are

$$z_1=\sqrt{3-2\sqrt{2}} \quad\text{and}\quad z_2=-z_1,$$

indicated in the figure. This means that

$$\int_{-\pi}^{\pi}\frac{d\theta}{1+\sin^2\theta}=2\pi i(B_1+B_2),$$

where

$$B_1=\operatorname*{Res}_{z=z_1}\frac{4iz}{z^4-6z^2+1}=\frac{4iz_1}{4z_1^3-12z_1}=\frac{i}{z_1^2-3}=\frac{i}{(3-2\sqrt{2})-3}=-\frac{i}{2\sqrt{2}}$$

and

$$B_2=\operatorname*{Res}_{z=-z_1}\frac{4iz}{z^4-6z^2+1}=\frac{-4iz_1}{-4z_1^3+12z_1}=\frac{i}{z_1^2-3}=-\frac{i}{2\sqrt{2}}.$$

Since

$$2\pi i(B_1+B_2)=2\pi i\left(-\frac{i}{\sqrt{2}}\right)=\frac{2\pi}{\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}}=\sqrt{2}\pi,$$

the desired result is

$$\int_{-\pi}^{\pi}\frac{d\theta}{1+\sin^2\theta}=\sqrt{2}\pi.$$

③ ~~$\frac{1+\cos 6\theta}{2}$~~

$$\frac{\cos^2 3\theta}{5-4\cos 2\theta} = \frac{1+\cos 6\theta}{2(5-4\cos 2\theta)}$$

If $z = e^{i\theta}$, $\frac{\cos^2 3\theta}{5-4\cos 2\theta} = \frac{1+\left(\frac{1}{z^6}+z^6\right)\frac{1}{2}}{2\left(5-2\left(\frac{1}{z^2}+z^2\right)\right)}$

$$= \frac{z^{12}+2z^6+1}{2^2 z^4(-2z^4+5z^2-2)}$$

The zeros of denominator inside the unit circle are

$z = \pm\frac{1}{\sqrt{2}}$,

$$\int_0^{2\pi} \frac{\cos^2 3\theta}{5-4\cos 2\theta}\,d\theta = \int_C \frac{z^{12}+2z^6+1}{2^2 z^4(-2z^4+5z^2-2)} \cdot \frac{dz}{iz}$$

let $f(z) = \frac{z^{12}+2z^6+1}{z^5(-2z^4+5z^2-2)}$, by long division,

$$f(z) = -\frac{z^8}{2} - \frac{5z^6}{4} - \frac{21}{8}z^4 \cdots$$

Thus, $\operatorname{Res}_{z=0} \frac{f(z)}{z^5} = \frac{-21}{8}$.

$\operatorname{Res}_{z=\pm\frac{1}{\sqrt{2}}} \frac{f(z)}{z^5} = \frac{27}{16}$

$$\therefore \int_0^{2\pi} \frac{\cos^2 3\theta}{5-4\cos 2\theta}\,d\theta = \frac{1}{4i}\cdot 2\pi i\left(\frac{12}{16}\right) = \frac{3\pi}{8}.$$

(4) let $z = e^{i\theta}$, then

$$\frac{1}{1 + a\cos\theta} = \frac{1}{1 + a\left(z + \frac{1}{z}\right)\frac{1}{2}}$$

$$= \frac{2z}{az^2 + 2z + a}$$

The zeros of denominator are $z = -\frac{1}{a} \pm \sqrt{\frac{1}{a^2} - 1}$

Since $|a| < 1$, the only pole inside the unit circle is

$$z_0 = -\frac{1}{a} + \sqrt{\frac{1}{a^2} - 1}$$

$$\int_0^{2\pi} \frac{d\theta}{1 + a\cos\theta} = \int_0^{2\pi} \frac{2z}{az^2 + 2z + a} \frac{dz}{iz}$$

$$= \int_C \frac{-2i}{az^2 + 2z + a} dz$$

$$= -2i(2\pi i) \operatorname*{Res}_{z = z_0} \left(\frac{1}{az^2 + 2z + a}\right)$$

$$= 4\pi \left(\frac{1}{2a\sqrt{\frac{1}{a^2} - 1}}\right)$$

$$= \frac{2\pi}{\sqrt{1 - a^2}}.$$